Encl (1)

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

AD-A267 965

| | |
|---|---|
| 1. REPORT NUMBER: Technical Note BN-1145 | 2. GOVT ACC |
| 4. TITLE (and Subtitle): On the A-Posteriori Estimation of the Modelling Error for the Heat Conduction in a Plate and Its Use for Adaptive Hierarchic Modelling | 5. TYPE OF REPORT & PERIOD COVERED: Final Life of Contract |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s): I. Babuska - I. Lee - C. Schwab | 8. CONTRACT OR GRANT NUMBER(s): 1 ONR N00014-J-1030; 2 AFOSR F49620-92-J-0100 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS: Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742-2431 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS: Department of the Navy, Office of Naval Research, Arlington, VA 22217 | 12. REPORT DATE: April 1993 |
| | 13. NUMBER OF PAGES: 26 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | 15. SECURITY CLASS. (of this report) |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

AUG 0 4 1993

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT

The paper addresses the a-posteriori error estimates of modelling heat convection problems in a plate. It derives estimators which are essentially local (of the size of the thickness of the plate). The estimator is guaranteed to be an upper bound. The lower bound of the error is also given. The asymptotic exactness of the estimator with respect to the thickness of the plate and with respect to the order of the model is proven. The adaptive procedure based on this a-posteriori error estimator is proposed. Numerical examples are given.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

/ 5

MARYLAND

COLLEGE PARK CAMPUS

# ON THE A-POSTERIORI ESTIMATION OF THE MODELLING ERROR FOR THE HEAT CONDUCTION IN A PLATE AND ITS USE FOR ADAPTIVE HIERARCHIC MODELLING

**by**

**I. Babuška**
**I. Lee**
**and**
**C. Schwab**

**Technical Note BN-1145**



**April 1993**

INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

On the a-posteriori estimation of the modelling error for the heat conduction in a plate and its use for adaptive hierarchic modelling

I. Babuška[1)], I. Lee[1)], C. Schwab[2)]

MD-93-03-BLS
TR93-03

[1)] Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742. Partially supported by the Office of Naval Research under Contract N00014-J1030.

[2)] Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, Maryland 21228-5398. Partially supported by the Air Force Office of Scientific Research under grant F49620-92-J-0100.

Abstract. The paper addresses the a-posteriori error estimates of modelling heat convection problems in a plate. It derives estimators which are essentially local (of the size of the thickness of the plate). The estimator is guaranteed to be an upper bound. The lower bound of the error is also given. The asymptotic exactness of the estimator with respect to the thickness of the plate and with respect to the order of the model is proven. The adaptive procedure based on this a-posteriori error estimator is proposed. Numerical examples are given.

## 1. Introduction

During recent years a large progress has been achieved in the field of the a posteriori error estimations and adaptive procedures in *the finite element method*. The basic ideas have been introduced in [1-3] and since then many error estimators appeared in the literature.

The basis of these estimators are essentially of the following types.

a) Residual estimators which employ the residuals of the finite element solution. For more about these we refer among others to [1-13]. These estimators often are utilizing the complementary energy principles (see e.g. [4,5,11,13]).

b) Flux-projection operators. These estimators are based on the idea of smoothing the fluxes (which are discontinuous) and comparison with the fluxes computed from the original finite element solution. For more we refer to [14-18].

c) Other categories include extrapolation estimators (see e.g. [19]) and interpolation-error bounds, see e.g. [20-21].

In [22-24] the comparison of the performance of various major error estimators and their robustness is given (see also there for an extensive list of references).

The a posteriori error estimators are the basis of the adaptive procedures. The indicators (which are the error estimators in the single elements) govern then the design of the mesh.

We underline that the error is understood here as the difference between the exact solution of the differential equation under consideration and its finite element approximation. The addressed differential equation describes the particular model, say, of the plate and the computed error characterizes the accuracy of the finite element method *with respect to*

*the exact solution of the selected plate model* and not with respect to the plate itself. Nevertheless the (exact) solution of the model (plate or shell) problem has to be understocd *as the approximate solution of the 3-dimensional formulation.*

Many models were proposed in the literature. We refer for example to the survey [25]. It is shown in [26] that various plate models could lead to significantly different results especially in the neighborhood of the boundary and in the presence of unsmooth input data. Hence the (exact) solution of the model problem has to be understood as an approximation of the (exact) solution of another "higher" problem.

In the example we mentioned, the "higher" problem is the three dimensional model.

Similarly as in any approximate method a hierarchy of models has to be available and be such that it allows to obtain the solution of the original "higher" model with a priori given accuracy. Practically this means that *an a-posteriori error estimator of the error of the model* in comparison with the exact solution of the higher model should exist and an adaptive procedure for the optimal model selection should be available.

In practice the exact solution of the model is impossible to find either. Hence we have to combine both errors--the error of the model and of the finite element solution. In the adaptive procedure both errors have to be controlled.

In contrast to the a posteriori error estimation techniques and adaptive approaches of the finite element method as described briefly earlier, the a posteriori error analysis of the models, and the problem of adaptive modelling has not been addressed in the literature until now. In this paper we address this problem for the solution of a model heat conduction problem in laminated

plates. Let us mention that this problem has also other physical interpretations. The theoretical results presented here are based on [27].

We concentrate in this paper on the error of the models, and its adaptive selection. We will only consider the case when the exact solution of the models is available. Hence we will assume that the error of the finite element solution is negligible in comparison with the model error.

The paper is organized as follows. After introducing some notation and the three-dimensional boundary value problem in Section 2, we put in Section 3 the hierarchical modelling in perspective with other approaches to derive reduced models. In Section 4 we collect basic properties of the hierarchical models from [27]. Section 5 contains the derivation of the a posteriori estimator for the modelling error and demonstrates its asymptotic exactness as $d \longrightarrow 0$, as well as its spectral exactness as $q \longrightarrow \infty$. In Section 6 we illustrate our theory by a simple numerical example, demonstrating in particular that our theoretical bounds of the effectivity indices are sharp. While Section 6 dealt with a smooth solution, we investigate in Section 7 the performance of our estimator in the presence of boundary layers. In Section 8 we address the adaptive selection of the model.

## 2. The basic notions and the problem formulation

By $\omega \subset \mathbb{R}^2$ we denote a bounded domain with a piecewise smooth boundary $\gamma$. For any $0 < d$ we define

$$\Omega = \omega \times \left(-\frac{d}{2}, \frac{d}{2}\right)$$

with the lateral boundary

$$\Gamma = \gamma \times \left(-\frac{d}{2}, \frac{d}{2}\right)$$

and the faces

$$R_\pm = \{(x_1,x_2,x_3) \,|\, (x_1,x_2) \in \omega,\ x_3 = \pm\, d/2\}.$$

Often we will write $\bar{x} = (x_1,x_2)$, $x = (x_1,x_2,x_3) = (\bar{x},x_3)$.

As a model problem we will consider the problem

$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \Gamma, \qquad (2.1)\\ D_n u &= f \quad \text{on} \quad R_\pm, \end{aligned}$$

where $D_n$ is the exterior unit normal derivative.

Let

$$H = \{u \in H^1(\Omega) \;|\; u = 0 \quad \text{on} \quad \Gamma\}$$

and define the bilinear form $B(\cdot,\cdot) : H\times H \longrightarrow \mathbb{R}$ and the functional $F(\cdot) : H \longrightarrow \mathbb{R}$

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad (2.2)$$

$$F(v) = \int_\Omega f(x_1,x_2)(v(x_1,x_2,\tfrac{d}{2}) + v(x_1,x_2, -\tfrac{d}{2}))d\bar{x}. \qquad (2.3)$$

Then the weak form of (2.1) reads:

Find $u \in H$ such that

$$B(u,v) = F(v), \ \forall v \in H. \qquad (2.4)$$

There exists a unique solution of the problem (2.2)-(2.4) provided that $f \in L_2(\omega)$ (the assumption on $f$ can be weakened). Further we define the energy norm

$$|\!|\!| u |\!|\!|^2 = B(u,u)^2. \qquad (2.5)$$

We remark that the problem (2.1)-(2.3) respectively (2.4) describes the heat conduction problem in the plate $\Omega$. We assumed that the material of the

plate is homogeneous; nevertheless the theory is valid with small changes for the laminated plates, too (see [27]).

## 3. The modelling problem

The problem of the modelling means here a reduction of the three dimensional problem (1.1) to a two dimensional one. There are three basic types of such reductions (which are available not only for the heat conduction problem but also for plates and shells). They are

1) Physical derivation,

2) Asymptotic derivation,

3) Hierarchical dimensional reduction.

Let us describe the main ideas of these three approaches.

1) Physical derivation

The problem (2.1) is first written in the form

$$\nabla u = \sigma \quad \text{in} \quad \Omega \tag{3.1a}$$

$$\operatorname{div} \sigma = 0 \quad \text{in} \quad \Omega \tag{3.1b}$$

$$\sigma \cdot n = f \quad \text{on} \quad R^{\pm} \tag{3.1c}$$

$$u = 0 \quad \text{on} \quad \Gamma. \tag{3.1d}$$

Here $\sigma$ defined in (3.1a) has the meaning of fluxes, (3.1b) and (3.1c) have the meaning of heat balance. For small $d$ we assume that the flux $\sigma_3$ is negligible so that the temperature $u$ through the thickness is constant. Then the heat balance through the thickness reads

$$d \operatorname{div} (\sigma_1, \sigma_2) = -2f.$$

This leads to

$$-\Delta_{\bar{x}} u = \frac{2f(\bar{x})}{d} \quad \text{in} \quad \Omega, \tag{3.2a}$$

$$u = 0 \quad \text{on} \quad \gamma. \tag{3.2b}$$

The second phase is to derive an improved model. We will follow the reasoning given in [29] (when applied to our problem).

Assume that the solution $\omega$ of (3.2) is available and denote it by $u_0$. Then the corresponding flux is

$$(3.3)\qquad (\sigma_1^{(0)},\sigma_2^{(0)}) = \nabla u_0.$$

The heat balance equation in $\Omega$ now reads

$$(3.4)\qquad \mathrm{div}_{\tilde{x}}\,\sigma + \frac{\partial\sigma_3}{\partial x_3} = 0$$

which suggests

$$(3.5)\qquad \frac{\partial\sigma_3}{\partial x_3} = -\mathrm{div}_{\tilde{x}}\,\sigma$$

$$(3.6)\qquad \sigma_3 = -x_3\,\mathrm{div}_{\tilde{x}}\,\sigma = -x_3\Delta_{\tilde{x}}u = 2\,\frac{f}{d}x_3.$$

$\sigma_3(x)$ satisfies then the originally prescribed boundary condition

$$(3.7)\qquad \sigma_3(x_1,x_2,\ \pm d) = f$$

and the improved flux is now $(\sigma_1,\sigma_2,\sigma_3)$. It satisfies the balance through the thickness and the boundary condition at $x_3 = \pm d/2$. Now using

$$(3.8)\qquad \frac{\partial u}{\partial x_3} = \sigma_3$$

we get the improved solution $\tilde{u}$

$$(3.9)\qquad \tilde{u}(x_1,x_2,x_3) = \frac{1}{d}\,f(x_1,x_2)\,x_3^2 + \psi(x_1,x_2).$$

Taking the heat balance through the thickness we get

$$d\,\Delta u = \frac{1}{12}\,d^3\Delta_{\tilde{x}}f + 2f + d\,\Delta_{\tilde{x}}\psi = 0$$

and hence

$$(3.10)\qquad -d\,\Delta_{\tilde{x}}\psi = 2f + \frac{1}{12}\,d^3\Delta_{\tilde{x}}f \quad \text{in} \quad \omega.$$

The boundary condition at $\gamma$ has to be understood in a weak sense $\int \tilde{u}(x_1, x_2, x_3)\, dx_3 = 0$ for $x_1, x_2 \in \gamma$ which yields

$$\psi(x_1, x_2) = -\frac{1}{12} d^2 f(x_1, x_2), \quad (x_1, x_2) \in \gamma$$

Now we can continue in the same vein and construct higher order approximations (in d) in addition to $\tilde{u}$ in (3.9).

The approaches of this type are used very often in the engineering literature. Nevertheless their relations to the original problems are not direct. In fact, in [30] p. 33 related to theories of this type for the plate problems one finds the statement "the refined theories do not yield reliable information from the standpoint of 3 dimensional problems. Nevertheless, these theories specify the principal stress of the plate".

2. Asymptotic derivation. For more about asymptotic analysis we refer e.g. to [31]. We will present here only the main idea.

First the problem (2.1) is scaled to unit thickness $d = 1$ by the substitution $\xi = (x_1, x_2)$, $\xi_3 = \frac{x_3}{d}$. Then the (2.1) takes the form

(3.11a)
$$-\Delta_\xi u - \frac{1}{d^2} \frac{\partial^2 u}{\partial \xi_3} = 0$$

with the boundary condition

(3.11b)
$$\frac{1}{d} \frac{\partial u}{\partial \xi_3} = \pm f \quad \text{for} \quad \xi_3 = \pm \frac{1}{2},$$

(3.11c)
$$u = 0 \qquad \text{for} \quad \xi \in \gamma.$$

We look for a solution in the form

(3.12)
$$u(\xi, \xi_3) = \frac{1}{d} u^{(0)}(\xi) + d\, u^{(1)}(\xi) + d^3 u^{(2)}(\xi) + \cdots .$$

Inserting (3.12) into (3.11) and matching powers of d, we get a sequence of

differential equations for $u^{(j)}$. The boundary condition (3.11c) at $\Gamma$ is generally not satisfied by (3.12) and must be accounted for by additional *boundary layers* of three-dimensional character.

The main problem with this and the previous approach is that they lead ultimately to the differentiation of the data $f$. It is well known that for unsmooth $f$ and fixed $d$ an increase of the number of terms in (3.12) leads to a decrease of the accuracy.

It is also obvious that neither one of the previous two methods creates a set (hierarchy) of models by which the exact solution of the three dimensional problem can be approached to any prescribed accuracy at fixed, positive thickness. Hence this type of modelling is not usable for our purposes.

3. The hierarchic modelling.

Denote by

(3.13) $$\mathcal{P} = \{\omega_i \mid \omega_i \subset \omega,\ 1 \le i \le n\}$$

a collection of $n$ domains with piecewise smooth boundaries $\partial\omega_i$ such that $\omega_i \cap \omega_j = 0$ if $i \neq j$ and $\bar{\omega} = \bigcup_{\ell=1}^{n} \bar{\omega}_j$ ($\mathcal{P}$ could be for example a triangulation of $\omega$).

For a vector of nonnegative integers

(3.14) $$q_n = (q_1, \ldots, q_n), \quad q_i \ge 0$$

and a sequence of linearly independent functions $\psi_j \in H^1(-1,1)$ we define the space

(3.15) $$S(\mathcal{P}, q) := \left\{ u \in H \;\middle|\; u|_{\omega_i} = \sum_{j=0}^{q_i} U_j^{(i)}(x)\psi_j\left(\frac{2x_3}{d}\right),\ \omega_i \in \mathcal{P} \right\}$$

Then $S(\mathcal{P},q) \subset H$ and the $(\mathcal{P},q)$-*model* is the boundary value problem:

Find $u(\mathcal{P},q) \in S(\mathcal{P},q)$ such that

$$B(u(\mathcal{P},q),v) = F(v), \quad \forall v \in S(\mathcal{P},q). \tag{3.16}$$

By this we obtain a system of elliptic equations which describe the $(\mathcal{P},q)$-model.

The essential question is how to select the functions $\psi_i$. This problem was analyzed in [28], [33]. For the laminated plates an analogous approach can be used. In [28], [33] it has been shown that a good choice is (because of the symmetry in $x_3$ of the solution)

$$\psi_j(\eta) = P_{2j}(\eta), \tag{3.17}$$

$P_\ell(\eta)$ is the $\ell$-th Legendre polynomial. In [33] the optimal choices of $\psi_j$ in dependence on various optimal'ty criteria were also analyzed.

## 4. The basic properties of the hierarchic modelling

Here we will address only the system (3.17) of functions and we will deal here only with the energy norm $|||u|||^2 = B(u,u)$. In [27] we analyzed also other norms. Denoting by $u$ the exact solution and by $u(\mathcal{P},q)$ the exact solution of the $(\mathcal{P},q)$-model, we define the modelling error

$$e(\mathcal{P},q) = u-u(\mathcal{P},q)$$

and will be interested in the a-posteriori estimation of $|||e|||$. In [27] the following theorem has been proven.

Theorem 4.1. Let $f \in H^{2N}(\omega)$ *and have compact support in* $\omega$. *Then for* $0 < d \le A$

$$|||e(\mathcal{P},q)||| \le C(A)d^{2N+1/2}\, \|f\|_{H^{2N}(\omega)}$$

*where*

$$N = \min q_i$$

*and* $H^k(\omega)$ *is the usual Sobolev space. The constant* $C$ *depends on* $A$ *and* $\omega$, *but is independent of* $d$ *and* $f$.

Let us underline that the assumption about the compact support of $f$ is almost essential. It can be weakened but certain boundary compatibility conditions have to be satisfied. Without these conditions the solution $u$ has a boundary layer and Theorem 4.1 is not valid; the overall accuracy as $d \to 0$ is at best $d^{1/2}$ in general. Nevertheless, in [34] it is shown that the low accuracy is confined only to a small neighborhood of $\gamma$. In addition, the full accuracy of $d^{2N+1/2}$ can be recovered by utilizing simultaneously different model orders $\sigma_i$ in the interior and near $\gamma$.

<u>Theorem 4.2</u>. *Assume that* $f \in H^{2N}(\omega)$ *and* $0 < d \le A$. *Let* $\mathcal{P}_t = \{\omega_1, \omega_2\}$ *where* $\omega_1 = \{(x_1,x_2) \in \omega \mid \mathrm{dist}\,(x_1,x_2) > t\}$, $\omega_2 = \omega\setminus\bar{\omega}_1$ *and let* $q_1 = N$, $q_2 = M \ge N$. *Then, for* $t > C_1 Nd|\ln d|$ *and* $M(N,d)$ *suitable,*

$$|\!|\!| e(\mathcal{P},q) |\!|\!| \le C(A) d^{2N+1/2} \|f\|_{H^{2N}(\omega)} .$$

*Here* $C_1$ *depends only on the Lipschitz constant of* $\gamma$.

Finally we have Theorem 4.3 which guarantees that we can select a sequence of models, the solution of which converges to the solution of the three dimensional problem (2.3). We have

<u>Theorem 4.3</u>. *Let* $d > 0$ *fixed,* $f \in L_2(\omega)$. *Then* $|\!|\!| e(\mathcal{P},q) |\!|\!| \to 0$ as $N = \min_{1\le i\le n} q_i \to \infty$.

The introduced hierarchy is a large family of models which allows to select the models having the following properties in the major special cases

a) If $f$ is smooth and $d$ is small then the particular model leads to an error which is the smallest possible with respect to $d$, i.e. we have asymptotic convergence with respect to $d \to 0$ with maximal possible rate, regardless of compatibility conditions violated by the data.

b) In all cases, for any error tolerance there exists a model in the

family which leads to the error in the prescribed range..

c) As will be seen later an effective a-posteriori error estimator of the modelling error exists and based on it an adaptive model selection procedure can be designed.

The following theorem plays an important role in the theoretical basis of the study of the adaptive hierarchal modelling.

Theorem 4.4. *For every* $(\mathcal{P}, q)$ *we have*

$$\int_{-d/2}^{d/2} e(x_1, x_2, x_3)\, dx_3 = 0 \qquad \text{a.e } \bar{x} \in \omega$$

*where we denoted* $e(\mathcal{P}, q) = e$.

We mention that in fact we have additional properties of $e$ which were utilized in [27].

## 5. The a-posteriori error estimation

Here we will give some basic results from [27] related to the a-posteriori estimation of the modelling error, when measured in the energy norm. The estimates in $L_2$-norm are also given in [27]. The a-posteriori error estimation will be basis for the adaptive modelling . For simplicity we will present here the results for $\mathcal{P} = \{\omega\}$ i.e. we assume that we will not partition $\omega$ and that $q_i = q_0$, $i = 1, \dots, n$.

In the sections 7 and 8, we will apply the results for the general case.

Let $0 < \varphi(x_1 x_2) \in W^{1,\infty}(\omega)$ and

$$Q := \max_{i=1,2} \left\| \frac{\partial \varphi^2}{\partial x_i} \Big/ \varphi^2 \right\|_{L^\infty(\omega)}. \tag{5.1}$$

We will be interested in the error measured in the weighted energy norm

(5.2)
$$\|e\|_{\varphi}^{2} = \int_{\Omega} |\nabla e|^{2} \varphi(x_1 x_2)\, dx_1 dx_2 dx_3.$$

A typical choice is for example

(5.3)
$$\varphi(x_1^0, x_2^0) = \exp \frac{\alpha}{d^{\rho}} (|x_1 - x_1^0| + |x_2 - x_2^{(0)}|)$$

where $\rho \in (0,1)$ and $0 < \alpha < \frac{\pi}{2}$. We can, of course, use other choices of $\varphi$, too. Assume now that we have computed the exact solution $u(q)$ of the model. Then with

(5.4)
$$r(x_1, x_2) = f - \frac{\partial u(q)}{\partial n} (x_1, x_2, d/2)$$

we define the error estimator $\mathcal{E}_q(u(q))$

(5.5)
$$\mathcal{E}_q(u(q)) = \sqrt{\frac{d}{2q+3}} \left[\int_{\omega} r^2 \varphi^2 dx_1 dx_2\right]^{1/2}.$$

To assess the quality of this estimator we will define the *effectivity index*

(5.6)
$$\theta_q(u(q)) = \frac{\mathcal{E}_q(u(q))}{\|u - u(q)\|_{\varphi}}.$$

We will say that $\mathcal{E}$ is an upper (lower) estimator if $\theta > 1$ ($\theta < 1$, respectively). Further we will call $\mathcal{E}$ $(\kappa_1, \kappa_2)$ *proper* with respect to a class $T$ of data $f$, if

(5.7)
$$0 \le \kappa_1 \le \theta_q \le \kappa_2 < \infty$$

holds for all data $f \in T$.

The estimator $\mathcal{E}_q$ is *asymptotically* resp. *spectrally exact* on the set of the data $T_d$ resp. $T_Q$ if $\theta_q \longrightarrow 1$ as $d \longrightarrow 0^+$ resp. $\theta_q \longrightarrow f$ as $q \longrightarrow \infty$.

Let us remark that as usual the asymptotic exactness requires considerably stricter assumptions on the data than the properness. Further, we underline that the quality of the estimators has to be understood relative to a class of data.

Let us finally remark that the weight can rapidly change in comparison with $d$ (see (5.3)). This makes the estimator very local and it can be used as basis of the adaptive modelling. In [27] we have proven the following theorem

Theorem 5.1.

a) *If* $f \in T = L_2(\omega)$ *then*

$$\theta \geq (1-\sqrt{2}\, dQ)\left(1 + \frac{\sqrt{2}}{\pi} dQ\left(1 + \frac{\sqrt{2}}{\pi} dQ\right)\right)^{-1/2} \tag{5.8}$$

b) *If* $f \in T_\beta := \{f \mid |||r|||_{1,\varphi} / |||r|||_{0,\varphi} \leq \beta < \infty\}$ (*where* $|||r|||^2_{k,\varphi} = \int_\Omega |\nabla^k_x r|^2 \varphi^2 dx_1 dx_2$) *then*

$$\theta \leq \left[1 + \frac{3}{2} \frac{d^2}{(2q+3)^2-4} (\beta^2 + Q^2)\right]^{1/2}. \tag{5.9}$$

*If, moreover,* $\varphi \equiv 1$, *then* $Q = 0$ *and the factor* $3/2$ *in* (5.9) *can be replaced by* $1/2$.

Theorem 5.1 shows that the estimator is $\kappa_1$, $\kappa_2$ proper with known $\kappa_1$ and $\kappa_2$. If we select the exponential weight $\varphi$ given in (5.3) then the estimator is asymptotically exact on $T_\beta$.

We see from the theorem that we get an upper estimator for a large class of data, namely we have only to assume that $f \in L_2(\omega)$. On the other hand, to get $\theta \sim 1$ we need more smoothness of the solution, for example that no boundary layer exists (this can be achieved for example if $f$ is smooth and has compact support similarly as in Theorem 4.1). Using in (5.8) and (5.9)

$Q \sim \frac{1}{d}$ we get a practically local estimator of high quality.

Theorem 5.1 concentrates on the case when $d \longrightarrow 0$. Let us now formulate the theorem when $d$ is fixed and $q \longrightarrow \infty$. In [27] we have proven

<u>Theorem 5.2</u>.

*a) If* $f \in T_\beta$ with $\beta = \bar{\beta}(q/d)^{1-\varepsilon}$, $\varepsilon > 0$, $\bar{\beta}$ *independent of* $d,q$ *then*

$$\theta(q) \le \left[1 + C_1\left(\frac{d}{q}\right)^{2\varepsilon}\right]^{1/2}, \quad i = 1,2$$

*where* $C_1$ *is independent of* $q$ *and* $d$.

*b) Let* $\lambda_k$, $\varphi_k(x_1,x_2)$ *denote the* k*-th eigenpair of* $-\Delta$ *in* $\omega$ *with* $\varphi_k|_\gamma = 0$. *Let further* $r(x_1,x_2) = \sum_k \rho_k\, \varphi_k(x_1,x_2)$ *and assume* $f$ *is such that* $\rho_1 \ne 0$ *and*

$$\sum_{k\ge2} \left[1 + \frac{d^2\lambda_k}{\pi^2}\right]^{-1} \theta_k \frac{|\rho_k|}{|\rho_1|} \le (\sqrt{2} - 1)\ (1 - q^{-1+\varepsilon})$$

*for some* $\varepsilon > 0$. *Then*

$$\theta(q) \ge (1 - C_2Q)(1 + 2C_2Q(1 + 2C_2Q))^{-1/2}$$

*where* $Q$ *is as in* (5.1) *and*

$$C_2 = \frac{d}{\sqrt{2}\ \{(2q+3)^2-4\}^{-\varepsilon}}$$

*and* $\theta$ *is the best constant in*

$$\|\varphi_k/\varphi_1\|_{L^\infty(\omega)} \le \theta_k.$$

In theorems 5.1 and 5.2 we analyzed only the error in energy norm. Nevertheless, similar results can also be obtained for the $L^2$-norm estimator derived in [27].

## <u>6. A simple example</u>

The results we have shown in the previous sections hold also in the case that $\omega = (-1,1)$. In the example here we use $\Omega = \omega \times (-d/2,\ d/2)$ and $f = \cos \frac{\pi}{2} x_1$. Then obviously

(6.1) $$u(x_1,x_2) = \cos\left(\frac{\pi}{2}x_1\right)\frac{\mathrm{ch}(\frac{\pi}{2}x_2)}{(\mathrm{sh}(\frac{\pi}{2}\frac{d}{2}))\frac{\pi}{2}}.$$

In this case we can compute (by computer algebra) the effectivity index $\theta_q$ explicitly and get

(6.2) $$\theta_q^2 = \frac{\mathcal{E}^2}{|||e|||^2} = 1 + \frac{d^2\pi}{m_q} + O(d^4).$$

The coefficients $m_q$ are listed in the Table 6.1.

Table 6.1. The coefficients $m_q$ in (6.2)

| $q$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $m_q$ | 240 | 360 | 936 | 1768 | 2856 | 4200 | 5800 |

We see that in fact the estimator is an upper one and that it is asymptotically exact.

In the Table 6.2 we report the values of $\theta_q$ computed directly. The finite element method with very fine mesh and high degree of elements $p$ was used to ensure that we obtained an accurate solution of the model. We selected $d = 1$ and $d = 2$, i.e. a very large thickness to show the effectiveness of the estimator. In the Table 6.7 we also report $\tilde{\theta}$

$$\tilde{\theta}^2 = 1 + \frac{d^2\pi}{m_q}$$

i.e. we neglected the term $O(d^4)$ in (6.2).

Finally we list the estimates based on (5.8) (5.9). In our case $r(x) = C\cos\frac{\pi}{2}x_1$ and hence $\beta = \frac{\pi}{2}$. In addition we report the relative energy error in %

Table 6.2. The effectivity indices and relative errors for large d.

| $q$ | $\theta(q)$ | $\tilde{\theta}(q)$ | Upper b | Lower b | $\|e\|\%$ | |
|---|---|---|---|---|---|---|
| 0 | 1.0200 | 1.0065 | 1.32 | 1 | 40.62 | $d = 1$ |
| 1 | 1.01324 | 1.0043 | 1.085 | 1 | 1.057 | |
| 2 | 1.0052 | 1.0033 | 1.040 | 1 | 0.0081 | |

| $q$ | $\theta(q)$ | $\tilde{\theta}(q)$ | Upper b | Lower b | $\|e\|\%$ | |
|---|---|---|---|---|---|---|
| 0 | 1.0742 | 1.025 | 1.626 | 1 | 64.51 | $d = 2$ |
| 1 | 1.0482 | 1.017 | 1.306 | 1 | 6.21 | |
| 2 | 1.0203 | 1.013 | 1.153 | 1 | 0.178 | |

Realize that $\theta(q_{i+1})/\theta(q_i) \approx \frac{m_i}{m_{i+1}}$. This is the effect of the spectral accuracy of the error estimator. The same effect is seen also in the improvement of the upper bound. This also shows that the term $O(d^4)$ neglected in (6.2) decreases as $q$ increases.

Let us underline that in the described example the thickness is no small (in fact when $d = 2$ then $\Omega$ is a square) and there is now boundary layer in the solution.

## 7. The problem in the presence of a boundary layer

Let us consider the case $f = 1$ where the solution has a boundary layer. Hence $f$ belongs to the class $T_\beta$ with a large $\beta$. Table 7.1 shows the effectivity index and the relative error for different $q$.

Table 7.1 The effectivity index and the error for the solution with the boundary layer

| d | d = 0.2 | | d = 1 | | d = 2 | |
|---|---|---|---|---|---|---|
| q | $\theta$ | $\|e\|\%$ | $\theta$ | $\|e\|\%$ | $\theta$ | $\|e\|\%$ |
| 0 | 1.011 | 9.88 | 1.063 | 42.53 | 1.140 | 87.65 |
| 1 | 1.46 | 0.505 | 1.463 | 5.14 | 1.454 | 12.291 |
| 2 | 1.778 | 0.197 | 1.778 | 2.01 | 1.778 | 4.736 |
| 3 | 1.933 | 0.112 | 1.933 | 1.14 | 1.933 | 2.681 |
| 4 | 1.911 | 0.079 | 1.908 | 0.803 | 1.912 | 1.883 |
| 5 | 1.760 | 0.064 | 1.743 | 0.655 | 1.760 | 1.529 |
| 6 | 1.554 | 0.056 | 1.511 | 0.587 | 1.554 | 1.358 |
| 7 | 1.348 | 0.053 | 1.292 | 0.557 | 1.348 | 1.269 |

We see that the effectivity index is growing with the value $q$, achieves maximum at $q \approx 3$ and then decreases. It is also interesting that the effectivity index is very insensitive to the thickness $d$. The behavior of $\theta$ can be explained as a combination of the influence of large $\beta$ and the spectral accuracy.

Let us now divide $\omega$ into subdomains. In our case $\omega = (-1,1)$ and we will divide $\omega$ into subdomains $I_i = (x_{i-1}, x_i)$, $i = 1,\dots 6$ and will assume $\varphi = \varphi_i$, $\varphi_i = 1$ on $I_i$, $\varphi_i = 0$ on $I_j$, $i \neq j$. This of course is not exactly in our framework of assumptions. Nevertheless we can understand this selection of $\varphi$ as an approximation. Hence we will speak about indicators instead of estimators. Let us consider the 6 domains defined by the sequence $\{x_i\} = \{-1, -0.75, -0.50, 0.0, 0.50, 0.75, 1.00\}$. The table 7.2 shows the indicators in these domains with uniform $q$. Because of symmetry we report the indicators on first three subdomains only.

Table 7.2. The error indicators for 6 domains

| | d = 0.2 | |
|---|---|---|
| $\omega_1$ | q = 1 | q = 2 |
| 1 | 0.576 -1 | 0.274 -1 |
| 2 | 0.359 -5 | 0.126 -5 |
| 3 | 0.224 -9 | 0.693 -9 |

We clearly see that the boundary layer is well indicated.

In Table 7.3 we show the value of the estimator in the first subdomain $I_1$ as function of $q$

Table 7.3. The error indicators in the subdomain $I_1$

| q | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\mathcal{E}(q)$ | 0.5761-1 | 0.2741-1 | 0.1687-1 | 0.1173-1 | 0.8768-2 | 0.6875-2 | 0.5578-2 |
| | | | | | | 8 | 9 |
| | | | | | | 0.4644-2 | 0.3945-2 |

Figure 5.1 shows the graph of the function $\mathcal{E}(q)$. We see $\mathcal{E}(q) \approx q^{-\beta}$ $1 < \beta < 2$. Let us mention that this rate is due to the combined effect of singularity in the corner (of order $r^2 \ell g\ r$) and the boundary layer.

Table 7.2 indicates that it is advantageous to use nonuniform $q$. In Tables 7.4a and b we show the error for different distributions of $q$ for sets of 18 and 20 subdomains. Because of symmetry we only list $q$ in the first 9 (10) domains.

We used the following two sets of subdomains defined by the meshes

a) 18 subdomains.

-1.000, -0.990, -0.975 -0.950 -0.925 -0.900 -0.850 -0.750

-0.500 0.00,

b) 20 subdomains

-1.000, -0.998 -0.990 -0.975 -0.950 -0.925 -0.900 -0.850

-0.750 -0.500 0.00

The thickness $d = 0.2$ and $f = 1$. We report in the Table 7.4a the error, the effectivity index and $W = \sum_{e=1}^{18} (q_e + 1)$ which (crudely) estimates the work.

Table 7.4a. The distribution of $q$ for 18 subdomains

| $\omega_i$ case | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 2 | 3 |
| 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 3 |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 4 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 5 | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 2 | 3 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 3 |
| Error % | 0.201 | 0.126 | 0.146 | 0.106 | 0.129 | 0.0947 | 0.110 | 0.089 | 0.197 | 0.112 |
| $\theta$ | 1.75 | 1.74 | 1.54 | 1.52 | 1.32 | 1.31 | 1.12 | 1.13 | 1.78 | 1.93 |
| W | 23 | 25 | 24 | 26 | 25 | 27 | 27 | 28 | 27 | 36 |

The Table 7.4b shows analogous results for the case of 20 subdomains. The first two subdomains are creating together the first subdomain in the case of 18 elements. This is indicated by a dashed line in the Table 7.4b.

Table 7.4b. The distributions of $q$ for 20 domains

| $\omega_i$ \ case | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 7 |
| 2 | 2 | 3 | 4 | 5 | 5 | 6 | 5 | 6 |
| 3 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 5 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 6 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Error % | 0.201 | 0.126 | 0.106 | 0.094 | 0.093 | 0.089 | 0.92 | 0.088 |
| $\theta$ | 1.75 | 1.74 | 1.52 | 1.31 | 1.21 | 1.13 | 1.11 | 1.04 |
| W | 23 | 25 | 26 | 27 | 27.5 | 28 | 28 | 28.5 |

In the Table 7.4b we define $\frac{1}{2} W = \frac{(q_1+1)+(q_i+1)}{2} + \sum_{\ell=1}^{10} (q_i + 1)$ to make a comparison possible. If the degrees in $\omega_1$ and $\omega_2$ are the same then we obtain identical results as in Table 7.4a.

We see from the Tables 7.4a,b that the optimal $q$ distribution depends on the error we wish to achieve and on the subdomains. The main error is in

the area of the boundary layer. Comparing the cases 6 and 7 in Table 7.4b we see that the case 6 leads to a smaller error than case 7. This underlines the importance of the right $q$ distribution.

We have shown in previous sections that the error estimator is not influenced too much by errors on the subdomains which are not too close to the given one (in the scale of $d$).

To show the sensitivity of the indicators with respect to the q-distribution, let us show in Table 7.5 the indicators in the subdomains $\omega_3, \omega_4, \omega_5$ (in the case of 18 subdomains) for some cases listed in Table 7.4a.

Table 7.5. Error estimators for the $q$ distribution given in Table 7.4

| Case | $\mathcal{E}_3$ | $\mathcal{E}_4$ | $\mathcal{E}_5$ |
|---|---|---|---|
| 1 | 0.104-2 | 0.782-3 | 0.867-3 |
| 2 | 0.162-2 | 0.719-3 | 0.866-3 |
| 4 | 0.163-2 | 0.719-3 | 0.866-3 |
| 6 | 0.163-2 | 0.719-3 | 0.866-3 |
| 9 | 0.105-2 | 0.871-3 | 0.487-3 |

Estimators $\mathcal{E}_6 - \mathcal{E}_9$ were completely independent of the q-distribution.

## 8. Problem of adaptive modelling

The goal of adaptive modelling is to design the distribution of $q_i$ so that for given error tolerance the work is minimal. The quantitative characterization of the work could be different. Here we will use the crude measure we introduced in the Section 7, namely $W = \sum(q_i + 1)$.

We will show one of the many possible adaptive strategies in two examples.

Example 1. Let us consider the problem we discussed in the previous section, namely $f = 1$ and $d = 0.2$. We will use the mesh which divides the domain into 18 subdomains. The target relative accuracy is $\tau = 0.1\%$.

First we solve the problem with $q = 1$ (uniform) and compute the error indicators on the particular domains. We single out the subdomains where the error is not necessarily made smaller. The principle is to add the smallest indicators $\varepsilon_i^2$ so that their error is $\frac{\tau^2}{5} B(u,u)$. We conclude that $q$ has to be increased on 8 subdomains, namely $\omega_1 - \omega_4$ and $\omega_{15} - \omega_{18}$. The goal is to increase $q_i$ on these domains so that we obtain the error in the prescribed tolerance. Using the fact that the indicator on a domain $\omega_\nu$ is not essentially influenced by the indicators on the other subdomains, we will "freeze" the solution on all subdomains except the one under consideration. By this procedure we obtain the following distribution of $q$: 9,6,4,2,1,1,1,1,1 (Because of the symmetry we report here and in what follows only the subdomains for $x < 0$.) By this distribution we achieved the relative error 0.077%. Comparing this distribution with Table 7.4a we see that the used strategy yielded the right distribution.

Example 2. Here we assume that

$$f = \frac{2x}{\left(x^2+\alpha^2\right)^4}$$

and that $d = 0.2$, $\alpha = 0.1$.

We will use 10 subdomains characterized by the mesh $x_2 = -1, -0.8, -0.6, -0.4, -0.2, -0.0, 0.2, 0.4, 0.6, 0.8, 1.0$, with the target accuracy 0.1%. In this case the main error is in the place where the solution is unsmooth, i.e. in the middle $(x \approx 0)$ of the domain. The adaptive procedure gave the following distribution of $q_i$ : 1, 1, 1, 2, 3. The relative energy error achieved was 0.0743%.

## REFERENCES


[1] Babuška, I., W.C. Rheinboldt, [1978]: Error estimates for adaptive finite element computations, SIAM J. Numer. Anal. 15(4), 736-754.

[2] Babuška, I., W.C. Rheinboldt, [1978]: A-posteriori error estimates for the finite element method, Int. J. Numer. Methods Engrg. 12, 1597-1615.

[3] Babuška, I., A. Miller, [1981]: A-posteriori error estimates and adaptive techniques for 'he finite element method, Tech. Note BN-968, Institute for Physical Science and Technology, Univ. of Maryland.

[4] Ladeveze, P., D. Leguillon, [1983]: Error estimation procedure in the finite element method and applications, SIAM J. Numer. Anal. 20, 485-509.

[5] Kelly, D.W., [1984]: The Self-Equilibration of residuals and complementary a-posteriori error estimates in the finite element method, Internat. J. Numer. Meth. Engrg. 20, 1491-1506.

[6] Bank, R.E., A. Weiser, [1985]: Some a-posteriori error estimates for elliptic partial differential equations, Math. Comp. 44, 283-301.

[7] Babuška, I., D. Yu, [1987]: Asymptotically exact a-posteriori error estimator for biquadratic elements, Finite Elements Anal. Design 3, 341-354.

[8] Oden, J.T., L. Demkowicz, W. Rachowicz, T.A. Westermann, [1989]: Toward a universal h-p adaptive finite element strategy: Part 2, A-posteriori error estimates, Comp. Methods Appl. Mech. Engrg. 77, 113-180.

[9] Duran, R., M.A. Muschietti, R. Rodriguez, [1991]: On the asymptotic exactness of the error estimator for linear triangular elements, Numer. Math. 59, 78-88.

[10] Verfürth, R., [1989]: A-posteriori error estimators for the Stokes equation, Numer. Math. 55, 309-325.

[11] Ainsworth, M., A. Craig, [1992]: A-posteriori error estimators in the finite element method, Numer. Math. 60, 429-463.

[12] Babuška, I., R. Duran, R. Rodriguez [1992]: Analysis of the efficiency of an a-posteriori error estimator for linear triangular finite elements, SIAM J. Numer. Anal. 29(4), 947-964.

[13] Ainsworth, M., J.T. Oden, [1993]: A unified approach to a-posteriori error estimation using element residual methods, Numer. Math., to appear.

[14] Zienkiewicz, O.C., J.Z. Zhu, [1987]: Simple error estimator and the adaptive procedure for practical engineering analysis, Internat. J. Numer. Methods Engrg. 3, 243-249.

[15] Shephard, M.S., Q. Niu, P.L. Bachmann, [1989]: Some results using stress projectors for error indication, in: J.E. Flaherty, P.J. Paslow, M.S. Shephard, J.K. Vasilakes, eds., Adaptive Methods for Partial Differential Equations (SIAM Philadelphia), 83-89.

[16] Wiberg, N.E., F. Abdulwahab, [1991]: A-posteriori error estimation based on superconvergent derivatives and equilibrium, Publ. 92:1, Department of Structural Mechanics, Chalmers University of Technology.

[17] Zienkiewicz, O.C., J.H. Zhu, [1992]: The superconvergent patch recovery and a-posteriori error estimates, Part 2: Error estimates and adaptivity, Internat. J. Numer. Methods Engrg., 1365-1382.

[18] Babuska, I., R. Rodriguez, [1993]: The problem of the selection of an a-posteriori error indicator based on smoothing techniques, Internat. J. Numer. Methods Engrg. 38, 539-567.

[19] Szabo, B.A., [1986]: Mesh design for the p-version of the finite element method, Comput. Methods Appl. Mech. Engrg. 55, 181-197.

[20] Demkowicz, L., P. Devloo, J.T. Oden, [1985]: On an h-type mesh-refinement strategy based on minimization of interpolation error, Comp. Methods Appl. Mech. Engrg. 53, 67-89.

[21] Eriksson, K., C. Johnson, [1988]: An adaptive finite element method for linear elliptic problems, Math. Comp. 50, 361-383.

[22] Struboulis, T., K.A. Hague, [1992]: Recent experiences with error estimation and adaptivity Part I: Review of error estimators for scalar elliptic problems, Comp. Methods Appl. Mech. Engrg. 97, 399-436.

[23] Struboulis, T., K.A. Hague, [1992]: Recent experiences with error estimation and adaptivity, Part II: Error estimation for h-adaptive approximations on grids of triangles and quadrilaterals, Comput. Methods Appl. Mech. Engrg. 100, 359-430.

[24] Babuška, I., T. Struboulis, C.S. Upadhyay, [1993]: A model study of the quality of a-posteriori error estimators for linear elliptic problems, Part I: Error estimation in the interior of patchwise uniform grids. To appear.

[25] Gileweski, W., M. Radwanska, [1991]: A survey of finite element models for the analysis of moderately thick shells. Finite Elements in Analysis and Design 9, 1-21.

[26] Babuška, I., L. Li, [1992]: The problem of plate modeling. Theoretical and computational results, Comp. Methods Appl. Mech. Engrg. 100, 249-273.

[27] Babuška, I., C. Schwab, [1993]: A-posteriori error estimates for hierarchical models of elliptic boundary value problems on thin domains. To appear.

[28] Vogelius, M., I. Babuška, [1981]: On a dimensional reduction method, Math. Comp. 37, 31-46, 47-68.

[29] Noor, A.K., J.M. Peters, [1988]: A-posteriori estimates for shear correction factors in multilayered composite cylinders, ASCE-Engrg. Mech. Division 115, 1225-1243.

[30] Ambartsumjan, S.A., [1991]: *Theory of anisotropic plates*, Hemisphere Publishing Corporation, New York, Washington (Originally published by Nauka, Moscow 1987).

[31] Ciarlet, P.G. [1990]: *Plates and junctions in elastic multi-structures, An asymptotic analysis*. Masson Publ., Paris and Springer, Berlin.

[32] Mazja, V.G., S.A. Nasarov, B.A. Plamenevskii, [1991]: Asymptotic theory of elliptic boundary value problems in singularly perturbed domains, Vol. 2 (in German), Akademie Verlag Berlin.

[33] Schwab, C., [1989]: Dimensional reduction for elliptic boundary value problems. Ph.D. thesis, University of Maryland, College Park, Maryland 20742.

[34] Schwab, C., [1992]: Boundary layer resolution in hierarchical plate modelling, Preprint TR92-13, Department of Mathematics and Statistics, University of Maryland, Baltimore County.

**The Laboratory for Numerical Analysis** is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.

To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

Further information may be obtained from **Professor I. Babuška**, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742-2431.